Scalar Product

Fact — The **scalar product** of two vectors **a**, **b** is denoted by $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ as well as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} = ap + bq + cr$$

Fact — Two vectors are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$

Example

Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$

Equation of a line

Example

Find the vector equation of the line $x = 2y = 7z$

Example

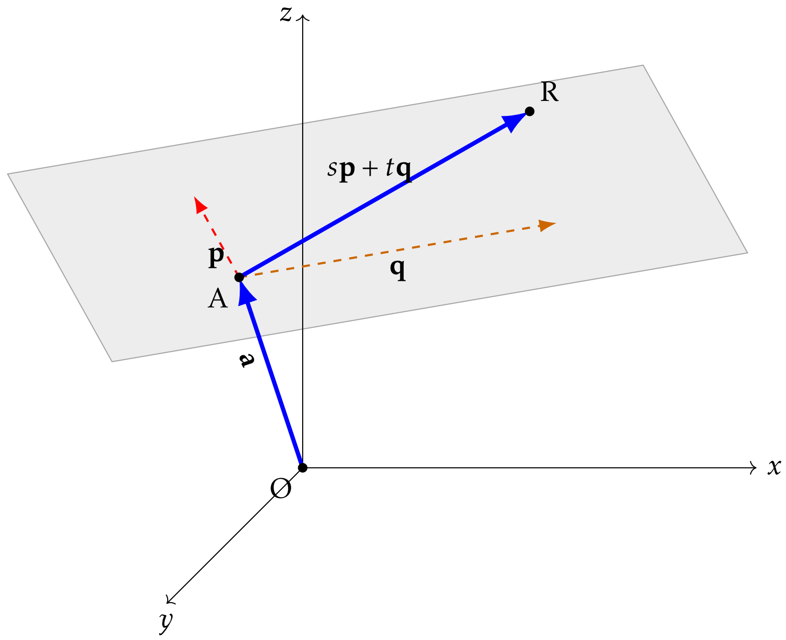
Find the vector equation of the line $\frac{x-1}{3} = \frac{y+1}{4} = \frac{z-3}{5}$

Example

Find the cartesian equation of the line $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

What happens if one the variables in the direction of the line is 0?

Equation of a plane



Fact — We can express every point on the plane through point $\mathbf{a}$, with non-zero, non-parallel vectors $\mathbf{p}, \mathbf{q}$ in the form $\mathbf{r} = \mathbf{a} + s\mathbf{p} + t\mathbf{q}$

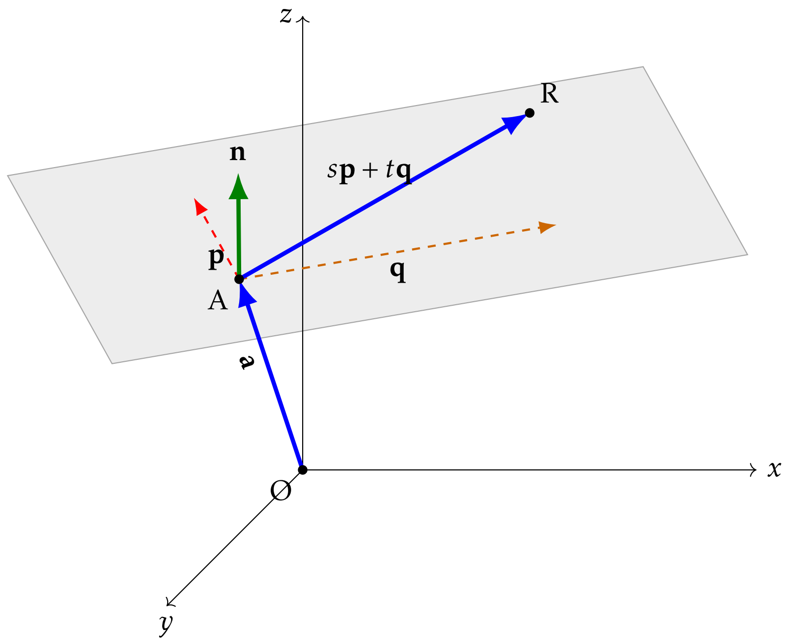
Example

Find a vector equation of the plane through the point $A(1,1,1)$, $B(1,-3,2)$ and $C(1,0,1)$

Example

Find a cartesian equation of the plane

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$



Fact — Three equations for a plane:

1. *Vector equation*: $\mathbf{r} = \mathbf{a} + s\mathbf{p} + t\mathbf{q}$
2. *Normal equation*: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
3. *Cartesian equation*: $px + qy + rz = k$

Example

Find the cartesian equation of the plane through the point $(1, 2, 3)$ with normal $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

Angles

Angle between two lines

Recall that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ where θ is the angle between the two vectors. So we can find the angle between two lines:

Example

Find the angle between the line joining $(1,2)$ and $(3,-5)$ and the line joining $(2,-3)$ to $(1,4)$.

Example

Find the angle between the line joining $(1,3,-2)$ and $(2,5,-1)$ and the line joining $(-1,4,3)$ to $(3,2,1)$:

Example

Find the angle between the diagonals of a cube.

Angle between a line and a plane

Example

Find the acute angle between the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{1}$ and the plane $x - y + z = 0$.

Angle between two planes

Example

A pyramid of height 3 units stands symmetrically on a rectangular base $ABCD$ with $AB = 2$ units and $BC = 4$ units. Find the angle between slanting faces

Vector product

Fact — The **vector product** of two vectors $\mathbf{a}$, $\mathbf{b}$ is given by:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{ab}$ in the right hand sense.

Example

Find the vector products

(a) $\mathbf{i} \times \mathbf{j}$

(b) $\mathbf{k} \times \mathbf{k}$

(c) $\mathbf{j} \times \mathbf{i}$

Fact — For vectors, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ and scalar s:

$$\begin{aligned} &\mathbf{p} \times \mathbf{q} = -\mathbf{q} \times \mathbf{p} \\ &s(\mathbf{p} \times \mathbf{q}) = (s\mathbf{p}) \times \mathbf{q} = \mathbf{p} \times (s\mathbf{q}) \\ &(\mathbf{p} + \mathbf{q}) \times \mathbf{r} = \mathbf{p} \times \mathbf{r} + \mathbf{q} \times \mathbf{r} \end{aligned}$$

Example

Compute $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Example

Find the vector product $\begin{pmatrix}1\\-3\\-5\end{pmatrix}\times\begin{pmatrix}0\\-1\\-2\end{pmatrix}$

Example

Find the area of the triangle ABC with vertices $A(1,2,3)$, $B(4,6,2)$ and $C(6,8,10)$.

Example

Find the normal equation of the plane

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Example

Find a vector equation for the line

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

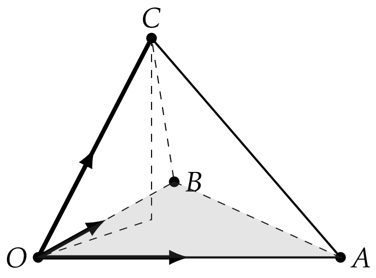
Fact — The vector equation of a line with direction vector **d** and point on the line **a** can be written as $(\mathbf{r} - \mathbf{a}) \times \mathbf{d} = 0$

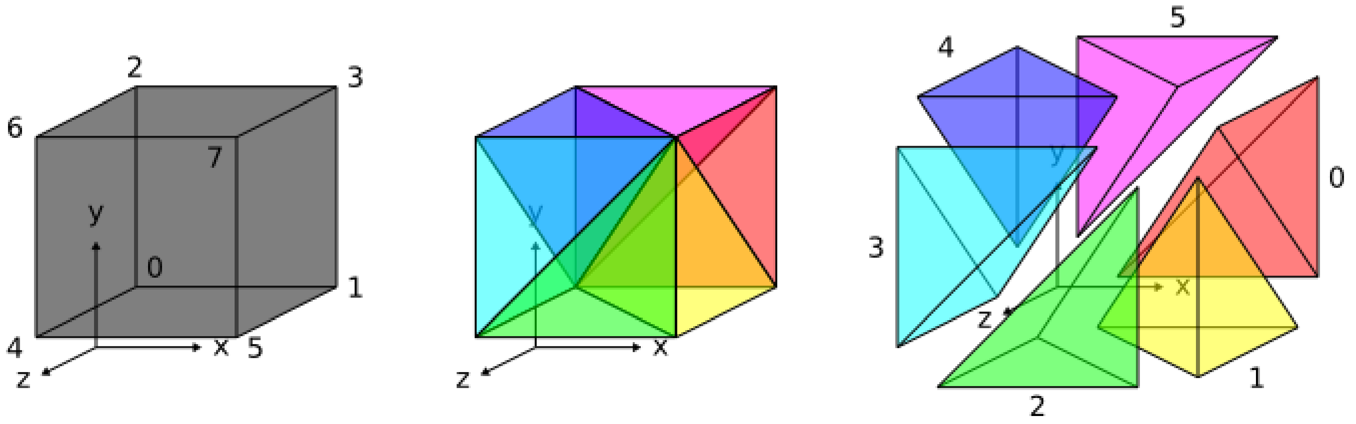
Scalar Triple Product

Example
What is the area of a triangle?

Example
What is the area of a parallelogram?

Example
What is the volume of a tetrahedron?





Example
What is the volume of a parallelepiped?

Definition. The **scalar triple product** of 3 vectors is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

Fact —

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

Example
Calculate the volume of a tetrahedron with vertices $P(1,3,2), Q(4,4,2), R(2,6,2)$ and $S = (3,5,7)$.

Fact — If we have $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Example

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & a \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$. Find the value of a for which it is singular

Intersections

Intersection of two lines

If we have two lines in two dimensions then

- They are parallel
  - They are the same line
  - They never meet
- They meet at a single point

If we have two lines in three (or more) dimensions then:

- They are parallel
  - They are the same line
  - They never meet
- They meed at a single point
- They are skew

Example

Find the cartesian equations of the line l joining $(-1,4,1)$ to $(3,6,2)$, and find whether this line intersects the line m with cartesian equation $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{-2}$

Intersection of three planes

Fact — $x + 2y + 3z = 6$ defines a plane, or in general $ax + by + cz = k$ defines a plane.

Therefore if we are looking for the intersection of three planes, we are looking for solutions to a system of simultaneous equations in 3 unknowns.

We have already seen that there are many different things which can happen when solving simultaneous equations in 3 variables:

- Unique solution
- No solutions
  - The 3 planes form a prism
  - The 3 planes are all parallel and not the same
- A line of solutions
  - The 3 planes form a sheaf (all distint planes around a point)
  - Two of the planes are the same and the other plane intersects it

Intersection of two planes

Similar to the intersection of three planes, except we cannot have a unique solution. They are either parallel or meet at a line.

Intersection of a line and plane

Example

Find the intersection between the given line and plane, or show they do not intersect

(a) $\mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$ and $2x - y + 2z = 5$

(b) $\frac{x-1}{-1} = \frac{y}{-3} = \frac{z+4}{2}$ and $x - 3y - 4z = 12$

Example

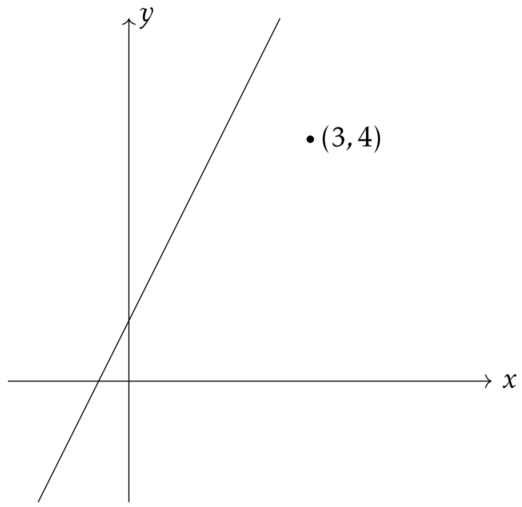
Show that the line $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ lies in the plane $x - 3y = 6$

Shortest distances

Shortest distance between a point and a line in 2D

Example

What is the shortest distance between the line $y = 2x + 1$ and the point $(3, 4)$?



Example

What is the shortest distance between the line $ax + by = c$ and the point (x_1, y_1)?

Shortest distance between a point and a line in 3D

Example

Find the shortest distance between the line $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and the point $P(1, 0, -3)$

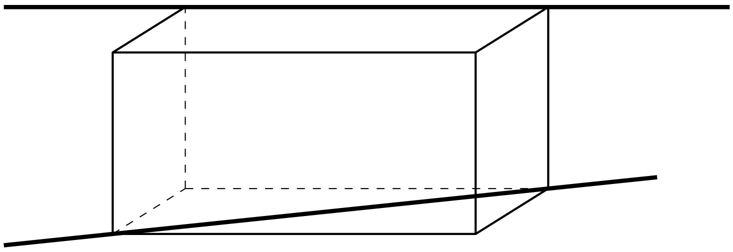
Shortest distance between a point and a plane

Example

Find the perpendicular distance of the point $P(4,5,6)$ from the plane $x+2y-2z=9$.

Shortest distance between two lines

Example
Find the shortest distance between the lines $\mathbf{r} = \mathbf{a}_1 + s\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + s\mathbf{b}_2$



Fact — The shortest distance between two lines, given by $\mathbf{r} = \mathbf{a}_1 + s\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + s\mathbf{b}_2$ is

$$d = \frac{|(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$